

Volume Expansion of Swiss-Cheese Universe

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Abstract

In order to investigate the effect of inhomogeneities on the volume expansion of the universe, we study modified Swiss-Cheese universe model. Since this model is an exact solution of Einstein equations, we can get an insight into non-linear dynamics of inhomogeneous universe from it. We find that inhomogeneities make the volume expansion slower than that of the background Einstein-de Sitter universe when those can be regarded as small fluctuations in the background universe. This result is consistent with the previous studies based on the second order perturbation analysis. On the other hand, if the inhomogeneities can not be treated as small perturbations, the volume expansion of the universe depends on the type of fluctuations. Although the volume expansion rate approaches to the background value asymptotically, the volume itself can be finally arbitrarily smaller than the background one and can be larger than that of the background but there is an upper bound on it.

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I. INTRODUCTION

The standard Big Bang scenario is based on an assumption of the homogeneous and isotropic distribution of matter and radiation. This assumption then leads to the Robertson-Walker spacetime geometry and the Friedmann-Lemaître (FL) universe model through the Einstein equations. This model has succeeded in explaining various important observational facts: Hubble's expansion law, the content of light elements and the isotropic cosmic microwave background radiation (CMBR)[1].

The CMBR conversely gives a strong observational basis for the assumption of homogeneity and isotropy of our universe by its highly isotropic distribution together with the Copernican principle; we know that our universe was highly isotropic and homogeneous at least on the last scattering surface where CMBR comes from[2]. Hence, in the early stage of our universe, the linear perturbation analysis in the FL universe model is a powerful tool to investigate the dynamical evolution of our universe[3].

In order to perform the perturbation analysis, we need an appropriate background universe model, i.e., information of the Hubble parameter, the density parameter and further the equation of state of the matter, radiation and so on, in the real universe. To fix the background universe, we use the observational data in the neighborhood of our galaxy. Especially the Hubble parameter is determined from the data about the distance-redshift relation within $100h^{-1}\text{Mpc}$ except for the type Ia supernova[4, 5]. However, the universe in a region within $100h^{-1}\text{Mpc}$ is highly inhomogeneous and hence there are non-trivial prescriptions to identify the present inhomogeneous universe to the homogeneous and isotropic FL universe model. If those procedure are not appropriate, we might miss finding the correct background universe.

It is often stated that the spatially averaged observational data in the vicinity of our galaxy are recognized as those of the background FL universe model. The Hubble parameter determined by the observed distance-redshift relation in our universe is regarded as the expansion rate of the volume of the region co-moving to matter. There are several researches for the effects of inhomogeneities on the volume expansion of the universe[6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. Especially, Nambu applied the renormalization group method to the second order cosmological perturbation theory and claimed that the expansion of the dust filled universe is decelerated by the inhomogeneities[12].

In the real universe, this back reaction effect might be very small[9]. However in order to get deeper insight into the dynamics of the inhomogeneous universe, we will consider the situation in which the back reaction of the inhomogeneities seems to be effective. For this purpose, we consider the Swiss-Cheese universe model and investigate the volume expansion in it. The original Swiss-Cheese universe model is constructed by choosing non-overlapping spherical regions in the background homogeneous and isotropic dust filled universe and then replacing these regions by the Schwarzschild space-time whose mass parameter is identical with the “gravitational” mass of the dust fluid in the removed region. On the other hand, in this article, we consider a modified version; we first remove spherical regions from the homogeneous and isotropic dust filled universe and then fill these regions with spherically symmetric but inhomogeneous dust balls. A spherically symmetric inhomogeneous dust ball is described by the Lemaître-Tolman-Bondi (LTB) solution which is an exact solution of the Einstein equations, and hence by this procedure, we obtain an exact solution of Einstein equations, which represents an inhomogeneous universe. Using this solution, we can study non-linear effects of inhomogeneities on the volume expansion of the universe without use of perturbation analysis.

In the LTB solution, shell crossing singularities are generic. Since the LTB solution is no longer valid after the occurrence of the shell crossing, we need to change the treatment if it occurs. As a crude approximation to describe the dynamics after the shell crossing, we adopt a model in which the shell crossing region is replaced by a spherical dust shell.

This article is organized as follows. In section II, we explain how to construct modified Swiss-Cheese universe models which are studied in this article. We investigate the volume expansion rate in the case of small perturbations in section III and in the highly inhomogeneous case in IV. In section IV, an alternative model is also constructed which describes the universe after the shell crossing and investigate the dynamics of this model. Finally, section V is devoted to summary and discussion.

We use the units in which $c = G = 1$ throughout the paper.

II. MODIFIED SWISS CHEESE UNIVERSE MODEL

In this section, we give a prescription to construct MSC universe model. First we consider a Einstein-de Sitter (EdS) universe and then remove spherical regions from it; these

removed regions should not overlap with each other. Next these regions filled with inhomogeneous dust balls with the same radii and the same gravitational mass as those of the removed homogeneous dust balls. In this MSC universe model, each inhomogeneous region is described by the Lemaître-Tolman-Bondi (LTB) solution which is an exact solution of Einstein equations.

LTB solution describes the dust filled spherically symmetric spacetime. Adopting synchronous and co-moving coordinate system, the line element is written as

$$\begin{aligned} ds^2 &= -dt^2 + \gamma_{ij}dx^i dx^j \\ &= -dt^2 + \frac{Y'^2(t, \chi)}{1 - \chi^2 k(\chi)} d\chi^2 + Y^2(t, \chi)(d\theta^2 + \sin^2 \theta d\varphi^2), \end{aligned} \quad (1)$$

where the prime ' denotes the differentiation with respect to the radial coordinate χ . In this coordinate system, components of 4-velocity u^a of a dust fluid element are

$$u^a = (1, 0, 0, 0). \quad (2)$$

The stress-energy tensor T_{ab} is then given by

$$T_{ab} = \rho(t, \chi) \delta_a^0 \delta_b^0, \quad (3)$$

where $\rho(t, \chi)$ is the rest mass density of the dust.

Einstein equations lead to the equations for the areal radius $Y(t, \chi)$ and the rest mass density $\rho(t, \chi)$ of the dust;

$$\dot{Y}^2 = -\chi^2 k(\chi) + \frac{2M(\chi)}{Y}, \quad (4)$$

$$\rho = \frac{M'(\chi)}{4\pi Y' Y^2}, \quad (5)$$

where $k(\chi)$ and $M(\chi)$ are arbitrary functions and the dot $\dot{}$ denotes the differentiation with respect to t .

We set $M(\chi)$ as

$$M(\chi) = \frac{4\pi\rho_0}{3}\chi^3, \quad (6)$$

where ρ_0 is a non-negative arbitrary constant. The above choice of $M(\chi)$ does not loose any generality. Eqs. (1)-(5) are invariant for the rescaling of the radial coordinate χ ,

$$\chi \rightarrow \tilde{\chi} = \tilde{\chi}(\chi). \quad (7)$$

Considering this property, we can choose above form of $M(\chi)$ as long as $\rho Y' > 0$.

The solutions of eq. (4) are given as follows:

In the region where $k(\chi) > 0$,

$$Y = \frac{4\pi\rho_0}{3k}(1 - \cos \eta)\chi, \quad (8)$$

$$t - t_0(\chi) = \frac{4\pi\rho_0}{3k^{3/2}}(\eta - \sin \eta); \quad (9)$$

in the region where $k(\chi) = 0$,

$$Y = \left[6\pi\rho_0\{t - t_0(\chi)\}^2\right]^{1/3}\chi; \quad (10)$$

in the region where $k(\chi) < 0$,

$$Y = \frac{4\pi\rho_0}{3|k|}(\cosh \eta - 1)\chi, \quad (11)$$

$$t - t_0(\chi) = \frac{4\pi\rho_0}{3|k|^{3/2}}(\sinh \eta - \eta), \quad (12)$$

where $t_0(\chi)$ is an arbitrary function. Note that $t_0(\chi)$ is the time when a shell focusing singularity appears, where ‘shell focusing singularity’ means $Y = 0$ for $\chi > 0$ and $Y' = 0$ at $\chi = 0$. In this article, we consider a region of $t > t_0$, and hence the time $t = t_0$ corresponds to the Big Bang singularity. Here, we focus on a case of $t_0 = 0$, i.e., simultaneous Big Bang. For simplicity, we consider the simplest version of MSC universe models shown in fig. 1; there is only one inhomogeneous spherical region at the center in each identical cubic region Ω . We focus on only one cubic co-moving region Ω (fig. 2). In this article, we consider a following model. Assuming $0 < \chi_1 < \chi_2 < \chi_3 < \chi_{sc}$,

$$k(\chi) = \begin{cases} k_0 & \text{for } 0 \leq \chi < \chi_1 \\ \frac{k_0}{2\chi^2} \left\{ \frac{(\chi^2 - \chi_2^2)^2}{\chi_1^2 - \chi_2^2} + \chi_1^2 + \chi_2^2 \right\} & \text{for } \chi_1 \leq \chi < \chi_2 \\ \frac{k_0}{2\chi^2} (\chi_1^2 + \chi_2^2) & \text{for } \chi_2 \leq \chi < \chi_3 \\ \frac{k_0}{2\chi^2} (\chi_1^2 + \chi_2^2) \left\{ \left(\frac{\chi^2 - \chi_3^2}{\chi_{sc}^2 - \chi_3^2} \right)^2 - 1 \right\}^2 & \text{for } \chi_3 \leq \chi < \chi_{sc} \end{cases}, \quad (13)$$

where k_0 is constant. In order to guarantee $1 - \chi^2 k > 0$, the following inequality should hold,

$$\kappa := \frac{k_0}{2}(\chi_1^2 + \chi_2^2) < 1. \quad (14)$$

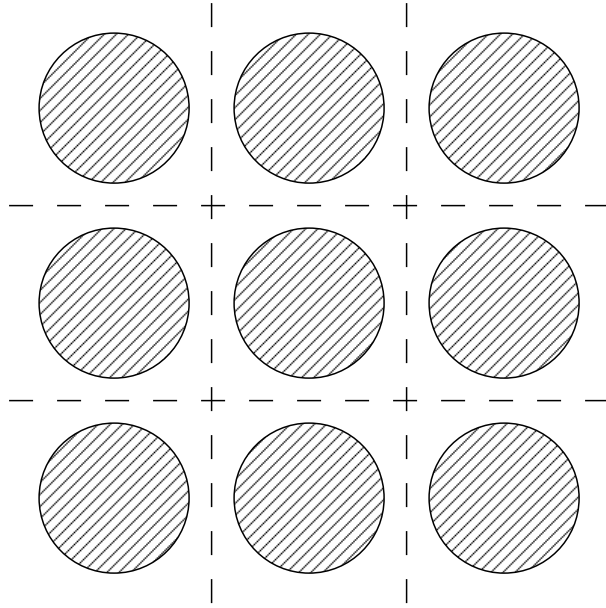


FIG. 1: The MSC universe model. Each shaded region represents the inhomogeneity and is described by LTB solution.

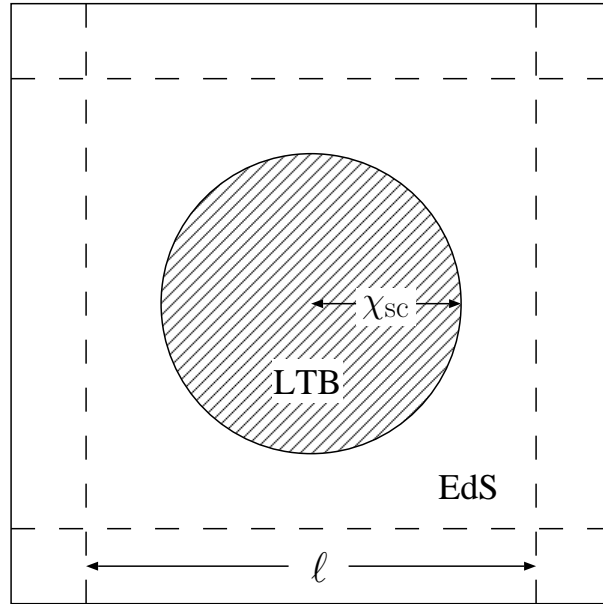


FIG. 2: One cubic region Ω of the MSC universe model. ℓ and χ_{sc} are the co-moving scales of this cubic region and the inhomogeneous region respectively.

We consider two cases; one is the case of $k_0 > 0$ and the other is the case of $k_0 < 0$. Then we investigate the volume expansion rate of a cubic co-moving region Ω . The volume V is defined by

$$V(t) := \int_{\Omega} \sqrt{\gamma} d^3x, \quad (15)$$

where γ is the determinant of the spatial metric γ_{ij} . The volume expansion rate is defined as \dot{V}/V .

III. THE CASE OF SMALL FLUCTUATIONS

In a region where

$$0 < \frac{9t|k|^{3/2}}{2\pi\rho_0} \ll 1$$

is satisfied, the areal radius $Y(t, \chi)$ is written in the form of power series as

$$Y(t, \chi) = a(t)\chi \left(1 - \frac{1}{20}\epsilon - \frac{3}{2800}\epsilon^2 \right) + O(\epsilon^3), \quad (16)$$

where

$$\epsilon(t, \chi) := \left(\frac{9t}{2\pi\rho_0} \right)^{2/3} k = \left(\frac{12t}{M(\chi)} \right)^{2/3} \chi^2 k, \quad (17)$$

and

$$a(t) := (6\pi\rho_0 t^2)^{1/3}. \quad (18)$$

Further, we consider the case of $|k|\chi^2 \ll 1$, which has been studied by Nambu by the second order perturbation analysis. The components of the metric tensor are written as

$$g_{\chi\chi} = a^2 \left[1 + \left(\frac{M}{12t} \right)^{2/3} \epsilon - \frac{1}{10} \frac{d}{d\chi}(\chi\epsilon) + O(\epsilon^2) \right], \quad (19)$$

$$g_{\theta\theta} = a^2 \chi^2 \left[1 - \frac{1}{10} \epsilon + O(\epsilon^2) \right], \quad (20)$$

$$g_{\varphi\varphi} = a^2 \chi^2 \sin^2 \theta \left[1 - \frac{1}{10} \epsilon + O(\epsilon^2) \right]. \quad (21)$$

From the above equations, it is easily seen that in the limit $\epsilon \rightarrow 0$ with t fixed, the metric tensor becomes that of EdS. Since the outside region is EdS universe, ϵ should vanish at the boundary $\chi = \chi_{\text{sc}}$ by the continuity of the metric tensor.

To compare our result with the study by Nambu[12], we impose conditions that spatial averages of the Cartesian components of the metric tensor and of the density agree with those of the *background* EdS universe up to the first order of ϵ .

The spatial average of a quantity F is defined as follows:

$$\langle F \rangle := \left(\int_{\Omega} d^3x \right)^{-1} \int_{\Omega} F d^3x. \quad (22)$$

We consider a Cartesian coordinate system (x, y, z) which is related to the spherical polar coordinate system (χ, θ, φ) in the ordinary manner as

$$\begin{aligned} x &= \chi \sin \theta \cos \varphi, \\ y &= \chi \sin \theta \sin \varphi, \\ z &= \chi \cos \theta. \end{aligned}$$

Then the spatial averages of Cartesian components of the spatial metric γ_{ij} are obtained as

$$\begin{aligned} \langle \gamma_{ij} \rangle &= a^2 \left[1 + \frac{4\pi}{3\ell^3} \int_0^{\chi_{\text{sc}}} \left\{ \left(\frac{M}{12t} \right)^{2/3} \epsilon \chi^2 - \frac{1}{10} \frac{d}{d\chi} (\chi^3 \epsilon) \right\} d\chi \right] \delta_{ij} + O(\epsilon^2) \\ &= (6\pi\rho_0 t^2)^{2/3} \left(1 + \frac{4\pi}{3\ell^3} \int_0^{\chi_{\text{sc}}} k \chi^4 d\chi \right) \delta_{ij} + O(\epsilon^2), \end{aligned} \quad (23)$$

where we have used $\epsilon = 0$ at $\chi = \chi_{\text{sc}}$. Therefore we should define the scale factor $a_{\text{B}}(t)$ of the background EdS universe as

$$a_{\text{B}}(t) := a(t) \left(1 + \frac{4\pi}{3\ell^3} \int_0^{\chi_{\text{sc}}} k \chi^4 d\chi \right)^{1/2}, \quad (24)$$

The above equation means that although the outside homogeneous region has the same geometry of EdS universe, it does not agree with *the background* EdS universe as long as

$$\int_0^{\chi_{\text{sc}}} k \chi^4 d\chi \neq 0. \quad (25)$$

The rest mass density ρ of the dust is written in the form of a power series with respect to ϵ as

$$\rho = \frac{1}{6\pi t^2} \left\{ 1 + \frac{1}{20\chi^2} \frac{d}{d\chi} (\chi^3 \epsilon) \right\} + O(\epsilon^2). \quad (26)$$

The spatial average of ρ is obtained as

$$\langle \rho \rangle = \frac{1}{6\pi t^2} + O(\epsilon^2). \quad (27)$$

Hence the background energy density ρ_B is defined by

$$\rho_B(t) := \frac{1}{6\pi t^2}. \quad (28)$$

Here note that the background Hubble equation

$$\left(\frac{\dot{a}_B}{a_B}\right)^2 = \frac{8\pi}{3}\rho_B, \quad (29)$$

holds.

By eq. (16), the 3-dimensional volume element $\sqrt{\gamma}$ is written as

$$\begin{aligned} \sqrt{\gamma} = a^3 \chi^2 \sin \theta & \left[1 - \frac{1}{20\chi^2} \frac{d}{d\chi}(\chi^3 \epsilon) + \frac{1}{2} \chi^2 k(\chi) + \frac{1}{700\chi^2} \frac{d}{d\chi}(\chi^3 \epsilon^2) \right. \\ & \left. - \frac{1}{40} k(\chi) \frac{d}{d\chi}(\chi^3 \epsilon) + \frac{3}{8} \chi^4 k^2(\chi) \right] + O(\epsilon^3). \end{aligned} \quad (30)$$

Using the above equation, the volume V defined by eq. (15) is obtained as

$$V(t) = a_B^3(t) (\ell^3 + V_1 + V_2 t^{2/3}) + O(\epsilon^3), \quad (31)$$

where

$$V_1 := \frac{3\pi}{2} \int_0^{\chi_{\text{sc}}} \chi^6 k^2 d\chi - \frac{2\pi^2}{3\ell^6} \left(\int_0^{\chi_{\text{sc}}} k \chi^4 d\chi \right)^2, \quad (32)$$

$$V_2 := -\frac{\pi}{20} \left(\frac{9}{2\pi\rho_0} \right)^{2/3} \int_0^{\chi_{\text{sc}}} k^2 \chi^4 d\chi < 0. \quad (33)$$

In eq. (31), $a_B^3 \ell^3$ is the 3-dimensional volume measured by background EdS geometry and extra terms come from inhomogeneities. These terms do not include first order perturbations, but come from second order perturbations.

The volume expansion rate is given by

$$\frac{\dot{V}}{V} = 3 \frac{\dot{a}_B}{a_B} + \frac{2V_2}{3\ell^3} t^{-1/3} + O(\epsilon^3). \quad (34)$$

The first term of the R.H.S. in the above equation corresponds to the background part. On the other hand, the second term implies that the back-reaction of inhomogeneities decelerates the volume expansion. It is worthwhile to note that this result does not depend on the detailed functional form of $k(\chi)$.

IV. THE CASE OF NON-LINEAR FLUCTUATIONS

In this section, we study the cases where $|k|\chi^2$ is not necessarily much smaller than unity. As in the case treated in the previous section, in order to specify an effect due to inhomogeneity, we need a background homogeneous cubic region to be compared with an inhomogeneous one. However the background homogeneous universe introduced in the previous section is not appropriate for the non-linear case; for example, too small k makes the background scale factor a_B defined in eq. (24) negative. In order to introduce an appropriate background, we consider the rest mass M_R defined by

$$\begin{aligned} M_R &:= \int_{\Omega} \rho u^0 \sqrt{-g} d^3x \\ &= 4\pi \int_0^{\chi_{\text{sc}}} \frac{\rho Y' Y^2}{\sqrt{1 - \chi^2 k}} d\chi + \rho_0 \left(\ell^3 - \frac{4\pi}{3} \chi_{\text{sc}}^3 \right) \\ &= \rho_0 \ell^3 \left\{ 1 + \frac{4\pi}{\ell^3} \int_0^{\chi_{\text{sc}}} \left(\frac{1}{\sqrt{1 - \chi^2 k}} - 1 \right) \chi^2 d\chi \right\}, \end{aligned} \quad (35)$$

where g is the determinant of the metric tensor of spacetime. M_R is a conserved quantity by virtue of the continuity of the rest mass density, $\partial_a(\rho u^a \sqrt{-g}) = 0$, where ∂_a is a partial derivative. We introduce a background as a cubic region with the same rest mass as the corresponding inhomogeneous cubic region. Note that in general, the rest mass of the dust M_R in a cubic co-moving region disagrees with that of the original EdS universe. Hence a cubic region of the original EdS universe is not background.

The rest mass density ρ_B and scale factor a_B of the background are introduced in the following manner,

$$\rho_B a_B^3 \ell^3 := M_R. \quad (36)$$

Even if we specify M_R , ρ_B and a_B are not fixed completely; we need one more condition. Here we impose a condition in which the volume $V(t)$ approaches to the volume $a_B^3(t)\ell^3$ of the background cubic region for $t \rightarrow 0$. In the limit of $t \rightarrow 0$, the volume V behaves as

$$\begin{aligned} V &= a^3 \ell^3 + 4\pi \int_0^{\chi_{\text{sc}}} \left(\frac{Y' Y^2}{\sqrt{1 - \chi^2 k}} - a^3 \chi^2 \right) d\chi \\ &\longrightarrow a^3 \ell^3 \left\{ 1 + \frac{4\pi}{\ell^3} \int_0^{\chi_{\text{sc}}} \left(\frac{1}{\sqrt{1 - \chi^2 k}} - 1 \right) \chi^2 d\chi \right\} \end{aligned} \quad (37)$$

Hence the background scale factor a_B is given by

$$a_B(t) := a \left\{ 1 + \frac{4\pi}{\ell^3} \int_0^{\chi_{\text{sc}}} \left(\frac{1}{\sqrt{1 - \chi^2 k}} - 1 \right) \chi^2 d\chi \right\}^{1/3}. \quad (38)$$

Here note that in the case of $\chi^2|k| \ll 1$, the above definition of a_B agrees with eq. (24) up to the first order of $\chi^2 k$. From eqs. (35), (36) and (38), we find that the background rest mass density ρ_B is completely the same as eq. (28). We can easily see that the background Hubble equation (29) also holds.

From eq. (5), we can easily see that if Y' vanishes, the rest mass density ρ becomes infinite and hence a singularity forms there. This is called shell crossing singularity. Hellaby and Lake showed that a necessary and sufficient condition for the appearance of a shell crossing singularity is [16]

$$k' > 0 \quad \text{for the region } k > 0, \quad (39)$$

$$(\chi^2 k)' > 0 \quad \text{for the region } k \leq 0. \quad (40)$$

In the case of $k_0 > 0$, the first condition does not hold and hence a shell crossing singularity does not appear. On the other hand, in the case of $k_0 < 0$, shell crossing singularities $Y' = 0$ necessarily appear since $(\chi^2 k)' > 0$ in the region $\chi_3 \leq \chi < \chi_{\text{sc}}$.

A. The Case of $k_0 > 0$

To estimate the volume V , we rewrite it in the form,

$$V = a^3 \left(\ell^3 - \frac{4\pi}{3} \chi_{\text{sc}}^3 \right) + 4\pi \int_0^{Y_{\text{sc}}} \frac{Y^2 dY}{\sqrt{1 - \chi^2 k}}, \quad (41)$$

where $Y_{\text{sc}} := a(t)\chi_{\text{sc}}$. From eq. (8), we can see that Y vanishes at $\eta = 2\pi$ by gravitational collapse, and hence by substituting $\eta = 2\pi$ into eq. (9), the singularity formation time $t = t_{\text{sg}}(\chi)$ is obtained as

$$t = t_{\text{sg}}(\chi) = \frac{8\pi^2 \rho_0}{3k^{3/2}}. \quad (42)$$

Denoting the inverse function of $t_{\text{sg}}(\chi)$ by $\chi_{\text{sg}}(t)$, the region of $0 \leq \chi \leq \chi_{\text{sg}}(t) < \chi_{\text{sc}}$ has already collapsed at time t larger than $8\pi^2 \rho_0 / 3k_0^{3/2}$. $\chi_{\text{sg}}(t)$ approaches to χ_{sc} for $t \rightarrow \infty$ asymptotically. Here note that in the integrand of eq. (41), $\chi = \chi_{\text{sg}}(t)$ at $Y = 0$ and $\chi = \chi_{\text{sc}}$

at $Y = Y_{\text{sc}}$. Since $\chi^2 k(\chi)$ is decreasing function with respect to χ in the region $\chi_3 < \chi < \chi_{\text{sc}}$ and vanishes just at $\chi = \chi_{\text{sc}}$, we find that $0 < \chi^2 k(\chi) \leq \chi_{\text{sg}}^2 k(\chi_{\text{sg}})$ holds in the integrand at sufficiently large t . This means that $\chi^2 k(\chi)$ also approaches to zero asymptotically, since $\chi_{\text{sg}}^2 k(\chi_{\text{sg}}) \rightarrow \chi_{\text{sc}}^2 k(\chi_{\text{sc}}) = 0$ for $t \rightarrow \infty$, and thus

$$V \longrightarrow a^3 \left(\ell^3 - \frac{4\pi}{3} \chi_{\text{sc}}^3 \right) + 4\pi \int_0^{Y_{\text{sc}}} Y^2 dY = a^3 \ell^3. \quad (43)$$

This equation means that the volume expansion rate approaches to the background value asymptotically, i.e.,

$$\frac{\dot{V}}{V} \longrightarrow 3 \frac{\dot{a}}{a} = 3 \frac{\dot{a}_{\text{B}}}{a_{\text{B}}} \quad \text{for } t \rightarrow \infty. \quad (44)$$

However the volume itself may approach to much different value from the background one $a_{\text{B}}^3 \ell^3$. $\sqrt{1 - \chi^2 k}$ can be made arbitrarily smaller than unity in the region $\chi_2 \leq \chi < \chi_3$; in the limit of $\kappa \rightarrow 1$, $\sqrt{1 - \chi^2 k} \rightarrow 0$ in this region (see eqs. (13) and (14)). Thus, if we set κ to be very close to unity, we obtain

$$a_{\text{B}} \sim a \left(\frac{4\pi}{\ell^3} \int_0^{\chi_{\text{sc}}} \frac{\chi^2 d\chi}{\sqrt{1 - \chi^2 k}} \right)^{1/3} \gg a. \quad (45)$$

In this case, the volume V approaches to the value much smaller than the background one, asymptotically. The volume expansion is also much different from that of the background in the intermediate stage (see fig. 3).

B. The Case of $k_0 < 0$

As mentioned in the above, shell crossing singularities $Y' = 0$ necessarily appear in this model. Before it, the volume expansion is shown in fig. 4. For $k(\chi) < 0$, background scale factor a_{B} cannot differ from original one very much. We plot the ratio of the volume $V(t)$ to $a^3 \ell^3$ and $a_{\text{B}}^3 \ell^3$. We find that the inhomogeneities decelerate the volume expansion before shell crossing singularity appears.

Here, we investigate the volume expansion rate after the appearance of this shell crossing singularity. The structure formed by the shell crossing depends on what is approximated by the dust matter. In case that the dust matter is extremely cold fluid, a shock wave will form after the shell crossing. If the dust matter consists of collisionless particles, a spherical

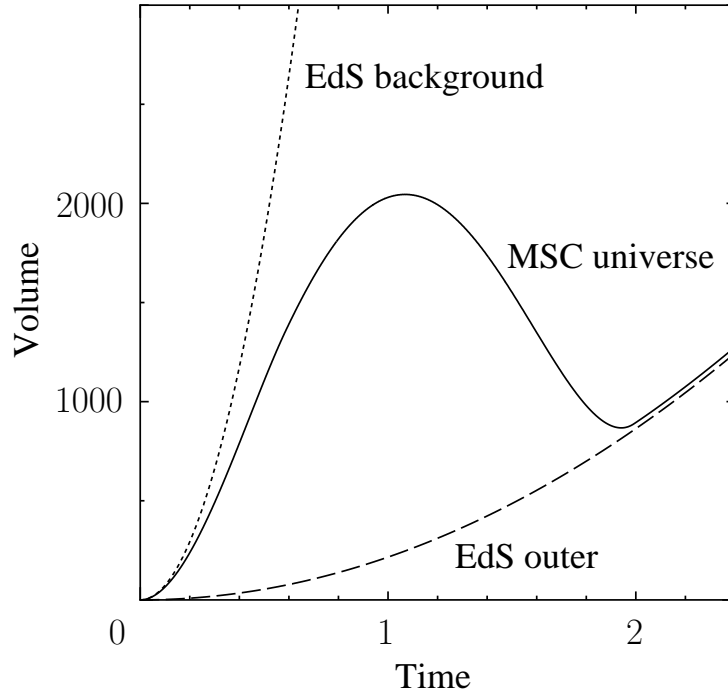


FIG. 3: Volume expansion with positive $k(\chi)$. The dotted line is the temporal evolution of the co-moving volume measured by the background EdS geometry and the dashed line is that measured by the outer EdS geometry. The time is set to unity, when the scale of the spherical inhomogeneous region $a(t)\chi_{\text{sc}}$ agrees with that of the horizon scale $a(t)/\dot{a}(t)$. The scale factor at $t = 1$ is also set to unity and the co-moving scale ℓ is set to $4\chi_{\text{sc}}$. Hence the co-moving volume measured by the outer EdS geometry at $t = 1$ is $\ell^3 (= 6^3 = 216)$.

wall will form. When the width of the shock wave or of the wall is much smaller than the radius of it, we can treat the shock front or the wall as a timelike singular hypersurface, where ‘timelike’ means that the unit normal vector n^a to the hypersurface is spacelike, i.e., $n^a n_a = 1$.

A timelike singular hypersurface is characterized by its surface-stress-energy tensor defined by

$$S_{ab} := \lim_{\varepsilon \rightarrow 0} \int_{-\varepsilon}^{\varepsilon} T_{cd} h_a^c h_b^d dx \quad (46)$$

where x is a Gaussian coordinate ($x = 0$ on the hypersurface) in the direction of the normal vector n^a , and $h_a^c := \delta_a^c - n_a n^c$ is a projection operator. A timelike singular hypersurface

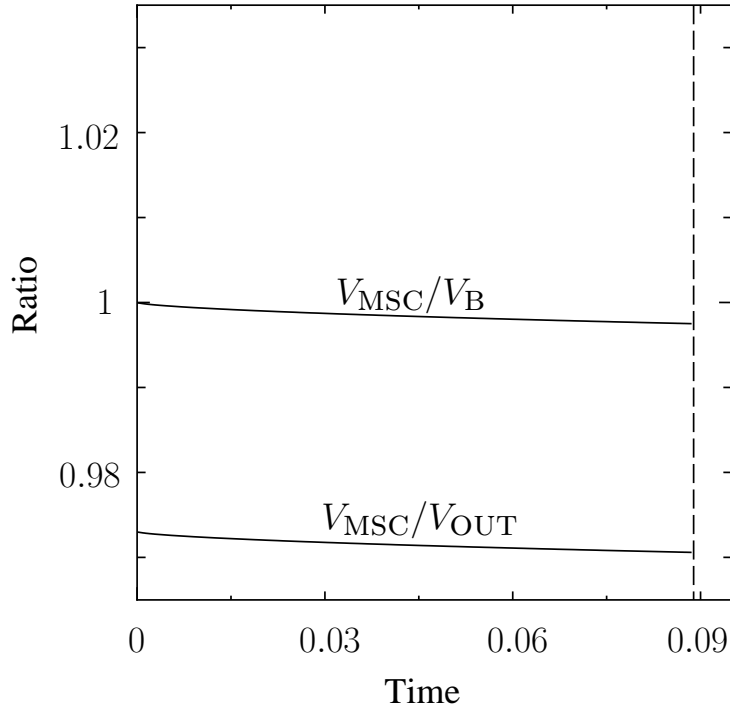


FIG. 4: Temporal evolution of the ratios before shell crossing singularity appears. The time is set to unity, when the scale of the spherical inhomogeneous region $a(t)\chi_{sc}$ agrees with that of the horizon scale $a(t)/\dot{a}(t)$. Dashed line corresponds to the shell crossing time. V_{MSC} is the volume of the co-moving cubic region Ω in modified Swiss Cheese universe. V_B is the volume of Ω measured by the background EdS geometry, i.e., $V_B = a_B^3 \ell^3$. V_{OUT} is the volume of Ω measured by the outer EdS geometry, i.e., $V_{OUT} = a^3 \ell^3$.

with a surface-stress-energy tensor of the form

$$S_{ab} = \sigma v_a v_b, \quad (47)$$

is called a world sheet generated by a trajectory of a dust shell, where σ is the surface-energy density and v_a is the 4-velocity of an infinitesimal surface element of the dust shell. Hereafter we focus on this case.

In order to get an insight into the dynamics after shell crossing, we study the volume expansion of a cubic region with a spherically symmetric dust shell. In the case of $k_0 < 0$, $k(\chi)$ is negative in the LTB region; $(\chi^2 k)' \leq 0$ in the inner region $0 \leq \chi \leq \chi_3$, while $(\chi^2 k)' > 0$ in the outer region $\chi_3 < \chi < \chi_{sc}$. In accordance with eq. (40), shell crossing necessarily occur in the outer region and we assume that this region collapses into a dust

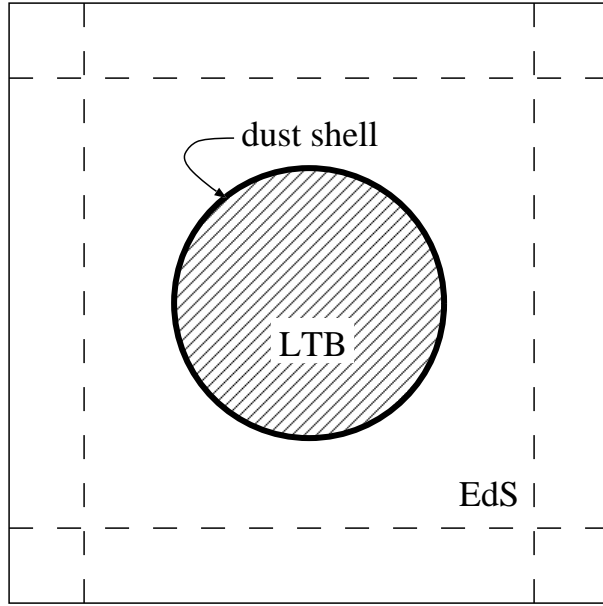


FIG. 5: The schematic configuration of the shell.

shell. Hence we focus on a situation in which $(\chi^2 k)' < 0$ holds inside the dust shell. The model is constructed by enclosing the interior LTB region by a spherically symmetric timelike singular hypersurface (see fig. 5).

The analysis by using the dust shell model works only before the dust shell reaches the boundary of the cubic co-moving region Ω . When the dust shell reaches the boundary, it collides with other dust shells centered in surrounding cubic regions. In this article, we do not consider the dynamics after collisions of the dust shells and assume that there is enough time before collisions of dust shells.

The interior region of the dust shell is described by the LTB solution with a line element,

$$ds_-^2 = -dt_-^2 + \frac{Y'^2(t_-, \chi_-)}{1 - \chi_-^2 k(\chi_-)} d\chi_-^2 + Y^2(t_-, \chi_-) (d\theta^2 + \sin^2 \theta d\varphi^2), \quad (48)$$

where the prime $'$ means the derivative with respect to χ_- . The equation for the areal radius Y is given by the same equation as eq. (4), i.e.,

$$\dot{Y}^2 = -\chi_-^2 k(\chi_-) + \frac{2M(\chi_-)}{Y}, \quad (49)$$

where the dot $\dot{}$ means the derivative with respect to t_- . Here we focus on the late time behavior of the volume expansion. In this case, since Y is monotonically increasing with respect to t_- , the “gravitational potential” $2M(\chi_-)/Y$ becomes much smaller than $-\chi_-^2 k(\chi_-)$

and hence we ignore this potential term in eq. (49). This approximation corresponds to an assumption in which the interior region of the dust shell is described by Minkowski geometry. Then the solution for Y is easily obtained as

$$Y(t_-, \chi_-) = \{-\chi_-^2 k(\chi_-)\}^{1/2} t_-. \quad (50)$$

The line element of the outer region is written as

$$ds_+^2 = -dt_+^2 + a^2(t_+) \{d\chi_+^2 + \chi_+^2 (d\theta^2 + \sin^2 \theta d\varphi^2)\}, \quad (51)$$

and Einstein equations reduce to the Hubble equation as

$$(\dot{a}\chi_+)^2 = \frac{M_+(\chi_+)}{a\chi_+}, \quad (52)$$

where $M_+(\chi_+)$ is related to the rest mass density ρ_+ of the outer region as

$$M_+(\chi_+) = \frac{4\pi}{3} \rho_+ a^3 \chi_+^3. \quad (53)$$

Note that by virtue of the spherical symmetry, angular coordinates, θ and φ , are common for both interior and exterior regions of the dust shell.

A physically and geometrically clear prescription to treat a timelike singular hypersurface has been presented by Israel[17]. In his prescription, junction conditions on metric tensor across the singular hypersurface lead to equations to determine the singular hypersurface itself, i.e., the equation of motion of a dust shell in our case. Using his prescription, dynamics of a vacuum void surrounded by a spherical dust shell in expanding universe has been analyzed by Maeda and Sato[18]. We can use their results since the situation considered here is completely the same as theirs.

By virtue of the spherical symmetry of the system considered here, the trajectory of the dust shell is given by

$$t_- = t_{s-}(t_+), \quad \chi_{\pm} = \chi_{s\pm}(t_+), \quad \theta = \text{constant} \quad \text{and} \quad \varphi = \text{constant}, \quad (54)$$

where the time coordinate t_+ in the exterior region has been adopted as an independent temporal variable. The areal radius R of the dust shell is then given by

$$R(t_+) := a(t_+) \chi_{s+} = Y_-(t_{s-}, \chi_{s-}) \quad (55)$$

Maeda and Sato derived a differential equation for the areal radius R of the dust shell as[18]

$$\frac{d^2 R}{dt_+^2} = \frac{1}{2R} \left\{ - (1 + VV_H + 2V^2 + V_H^2) + (1 - 4V^2)(1 + 2VV_H + V_H^2)^{1/2} \right\}, \quad (56)$$

where

$$H := \frac{\dot{a}(t_+)}{a(t_+)}, \quad (57)$$

$$V_H := HR, \quad (58)$$

$$V := \frac{dR}{dt_+} - V_H. \quad (59)$$

Using a solution of eq. (56), the coordinate radius χ_{s+} of the dust shell in the outer region is given by

$$\chi_{s+} = \frac{R}{a(t_+)}. \quad (60)$$

Equations for t_{s-} and χ_{s-} on the dust shell are given by

$$\begin{aligned} \frac{dt_{s-}}{dt_+} = & \left\{ 1 - \chi_{s-}^2 k(\chi_{s-}) \right\}^{1/2} \left\{ 1 - a^2(t_+) \left(\frac{d\chi_{s+}}{dt_+} \right)^2 + \left(\frac{dR}{dt_+} \right)^2 \right\}^{1/2} \\ & - \left\{ -\chi_{s-}^2 k(\chi_{s-}) \right\}^{1/2} \frac{dR}{dt_+}, \end{aligned} \quad (61)$$

$$\frac{d\chi_{s-}}{dt_+} = \frac{1}{Y'_-(t_{s-}, \chi_{s-})} \left(\frac{dR}{dt_+} - \dot{Y}_-(t_{s-}, \chi_{s-}) \frac{dt_{s-}}{dt_+} \right), \quad (62)$$

where the dot $\dot{}$ and the prime $'$ are derivatives with respect to t_- and χ_- , respectively. The volume V_{in} inside the dust shell is written as

$$\begin{aligned} V_{\text{in}} &= 4\pi \int_0^{\chi_{s-}} \frac{Y'Y^2}{\sqrt{1 - \chi^2 k(\chi)}} d\chi = 4\pi \int_0^R \frac{Y^2}{\sqrt{1 + Y^2/t_{s-}^2}} dY \\ &= 2\pi t_{s-}^3 \left[\frac{R}{t_{s-}} \sqrt{1 + \left(\frac{R}{t_{s-}} \right)^2} - \ln \left\{ \frac{R}{t_{s-}} + \sqrt{1 + \left(\frac{R}{t_{s-}} \right)^2} \right\} \right], \end{aligned} \quad (63)$$

where we have used eq. (50) in the second equality to estimate $\chi^2 k(\chi)$ in the integrand. We consider a normalized volume \tilde{V}_{in} by the volume of the removed homogeneous dust ball in the original EdS universe,

$$\tilde{V}_{\text{in}} := \frac{V_{\text{in}}}{4\pi R^3/3} = \frac{3}{2} \left(\frac{t_{s-}}{R} \right)^3 \left[\frac{R}{t_{s-}} \sqrt{1 + \left(\frac{R}{t_{s-}} \right)^2} - \ln \left\{ \frac{R}{t_{s-}} + \sqrt{1 + \left(\frac{R}{t_{s-}} \right)^2} \right\} \right]. \quad (64)$$

\tilde{V}_{in} is monotonically decreasing function of R/t_{s-} ; it approaches to unity in the limit of $R/t_{s-} \rightarrow 0$, while it vanishes in the limit $R/t_{s-} \rightarrow \infty$ (see fig. 6). In order to see temporal behavior of \tilde{V}_{in} , we need to solve eqs. (56) and (61).

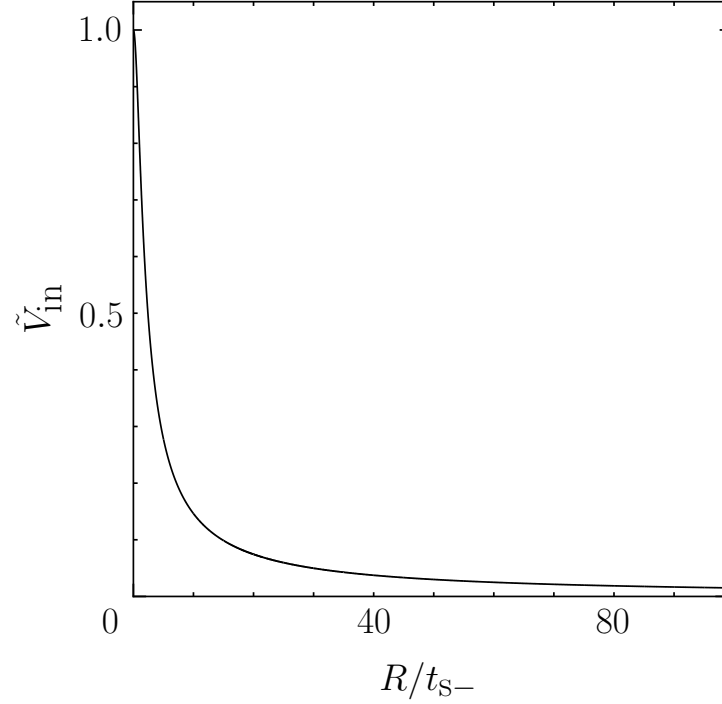


FIG. 6: Behavior of \tilde{V}_{in} as a function of R/t_{s-} .

For sufficiently large t_+ , the areal radius R of the dust shell becomes much smaller than the cosmological horizon scale H^{-1} . In this case, the motion of the dust shell is well described by the Newtonian approximation. Maeda and Sato showed that for sufficiently large t_+ , R behaves as[18]

$$R(t_+) \propto t_+^{(15+\sqrt{17})/24} \sim t_+^{0.797} \sim t_{s-}^{0.797}. \quad (65)$$

Hence we find that $R/t_{s-} \propto t_{s-}^{-0.203} \rightarrow 0$ for $t_{s-} \rightarrow \infty$. Using this result and eq. (64), we find $\tilde{V}_{\text{in}} \rightarrow 1$ for $t_{s-} \rightarrow \infty$, and hence for $t_+ \rightarrow \infty$,

$$V \rightarrow a^3(t_+) \ell^3. \quad (66)$$

This equation means that the volume expansion rate approaches to the background value asymptotically, i.e.,

$$\frac{\dot{V}}{V} \rightarrow 3 \frac{\dot{a}}{a} = 3 \frac{\dot{a}_{\text{B}}}{a_{\text{B}}} \text{ for } t \rightarrow \infty. \quad (67)$$

The effect of inhomogeneities on the volume expansion rate vanishes after the dust shell becomes much smaller than the horizon scale H^{-1} (see fig. 7). However, as in the case of $k_0 > 0$, the volume itself is different from the background value $a_B^3 \ell^3$. By eq. (38), we find that the asymptotic value of V is larger than the background value $a_B^3 \ell^3$. However it should

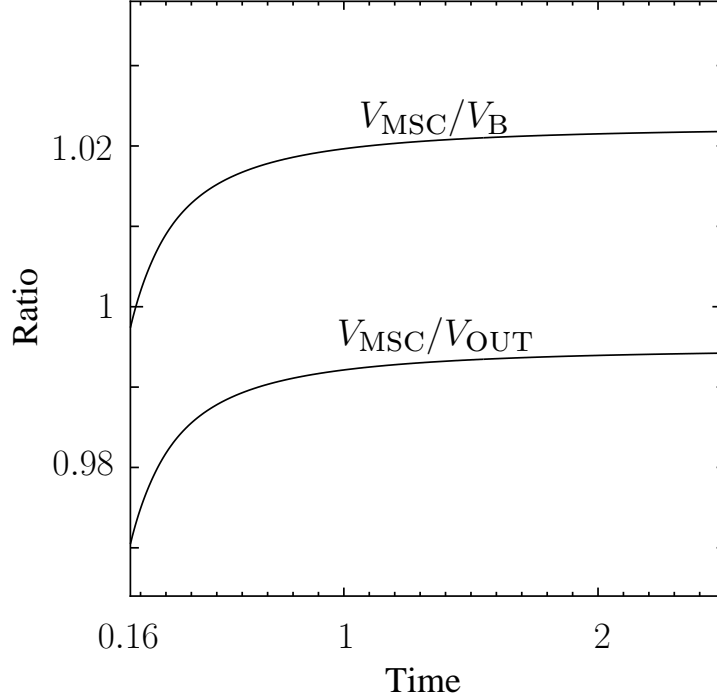


FIG. 7: Temporal evolution of the ratios after shell crossing singularity appears. The time is set to unity, when the scale of spherical inhomogeneous region $a(t)\chi_{\text{sc}}$ agrees with that of the horizon scale $a(t)/\dot{a}(t)$. V_{MSC} is the volume of the co-moving cubic region Ω in modified Swiss Cheese universe. V_B is the volume of Ω measured by the background EdS geometry, i.e., $V_B = a_B^3 \ell^3$. V_{OUT} is the volume of Ω measured by the outer EdS geometry, i.e., $V_{\text{OUT}} = a^3 \ell^3$.

be noted that there is an upper limit on the asymptotic value of V . Since k can be arbitrarily small in this model, $\sqrt{1 - \chi^2 k}$ can be made arbitrarily larger than unity except at $\chi = 0$ and $\chi = \chi_{\text{sc}}$. Hence we obtain

$$a_B(t_+) > a(t_+) \left(1 - \frac{4\pi}{3\ell^3} \chi_{\text{sc}}^3\right)^{1/3} > a(t_+) \left(1 - \frac{\pi}{6}\right)^{1/3}, \quad (68)$$

where the last inequality is obtained by setting $\chi_{\text{sc}} = \ell/2$. Using this inequality, we obtain

$$a_B^3(t_+) \ell^3 < a^3(t_+) \ell^3 < \left(1 - \frac{\pi}{6}\right)^{-1} a_B^3(t_+) \ell^3 \sim 2.10 \times a_B^3(t_+) \ell^3. \quad (69)$$

In contrast with the case of $k_0 > 0$, the volume itself can not be so different from the background value.

V. SUMMARY AND DISCUSSION

We have investigated an effect of inhomogeneities on the volume expansion in modified Swiss-Cheese universe model. We considered two cases; the inhomogeneities collapse into black holes ($k_0 > 0$), while inhomogeneities expands faster than the background volume expansion ($k_0 < 0$). When inhomogeneities can be treated as perturbations of Einstein-de Sitter universe, the volume expansion is decelerated due to the second order contribution of the perturbations in both models. This result agrees with Nambu's second order perturbation analysis.

Although the choice of background homogeneous universe is straightforward in the case of $|k|\chi^2 \ll 1$, it is not in the case of non-linear situation. We introduced the background homogeneous universe in order to satisfy the conditions; the cubic region in the background homogeneous universe with the same rest mass have the same evolution of the volume as that of corresponding region in the modified Swiss-Cheese universe in the limit of $t \rightarrow 0$.

In the case of non-linear fluctuation with $k_0 > 0$, we find that the volume expansion rate approaches to that of background universe asymptotically. Since the modified Swiss-Cheese model is an exact solution of Einstein equation, we can obtain the precise behavior of the volume expansion. We set it so that for $t \rightarrow 0$, the temporal evolution agrees with background one. Then we found that for $t \rightarrow \infty$, the temporal evolution agrees with that of outer EdS universe. We can see these asymptotic behaviors analytically but we have to rely only on the numerical method to obtain the behavior of the intermediate stages. From the result of the numerical calculation (fig. 3), we found that volume expansion is decelerated by the inhomogeneities. This behavior coincides with a result obtained by Nambu. We note that his result is based on the perturbation theory but our result is not. In our highly non-linear example, the volume expansion rate becomes negative at the intermediate stage. This result may be the characteristics of the effect of non-linear fluctuations which can not be treated by the perturbation method.

In the case of $k_0 < 0$, the shell crossing singularity appears in the inhomogeneous region. In this article, we assumed that a spherical dust shell forms after the shell crossing singu-

larity appears. We find that the inhomogeneities decelerate the volume expansion before it appears (fig. 4). This is consistent with the previous results by Nambu but the inhomogeneities cannot be treated as the perturbation of homogeneous universe. After a spherical dust shell forms, the volume expansion rate approaches to that of the background universe asymptotically. Our dust shell universe model (fig. 5) is a crude approximation. Therefore the behavior of the volume expansion obtained (fig. 7), especially at the early stage after the shell crossing, contains the influences of this crudeness. But the asymptotic behavior ($t \rightarrow \infty$) might be free from this approximation.

We fixed the form of $k(\chi)$. Temporal evolution of the volume depends on the form of $k(\chi)$ but the asymptotic behavior may be independent of the form of $k(\chi)$;

$$V \longrightarrow a^3 \ell^3 \quad \text{for } t \rightarrow \infty.$$

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